How to compute the rank of a Delaunay polytope

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Abstract

Roughly speaking, the rank of a Delaunay polytope (first introduced in [2]) is its number of degrees of freedom. In [3], a method for computing the rank of a Delaunay polytope P using the hypermetrics related to P is given. Here a simpler more efficient method, which uses affine dependencies instead of hypermetrics is given. This method is applied to classical Delaunay polytopes.

Then, we give an example of a Delaunay polytope, which does not have any affine basis.

1 Introduction

A lattice L is a set of the form $v_1\mathbb{Z} + \cdots + v_n\mathbb{Z} \subset \mathbb{R}^n$. A *Delaunay polytope* P is inscribed into an *empty sphere* S such that no point of L is inside S and the vertex-set of P is $L \cap S$. The Delaunay polytopes of L form a partition of $\mathbb{R}^n$.

The vertex-set $V = V(P)$ of a Delaunay polytope P is the support of a distance space (V, d_P). For $u, v \in V(P)$, the distance $d_P(u, v) = \|u - v\|^2$ is the *Euclidean norm* of the vector $u - v$. A *distance vector* $(d(v, v'))$ with $v, v' \in V$ is called a *hypermetric* on the set V if it satisfies $d(v, v') = d(v', v)$, $d(v, v) = 0$ and the following *hypermetric inequalities*:

$$H(b)d = \sum_{v,v' \in V} b_v b_{v'} d(v, v') \leq 0 \text{ for any } b = (b_v) \in \mathbb{Z}^V \text{ with } \sum_{v \in V} b_v = 1 \ . \tag{1}$$

The set of distance vectors, satisfying (1) is called the *hypermetric cone* and denoted by $HYP(V)$.

The distance d_P is a hypermetric, i.e., it belongs to the hypermetric cone $HYP(V)$. The *rank* of P is the dimension of the minimal by inclusion face F_P of $HYP(V)$ which contains d_P.

It is shown in [3] that d_P determines uniquely the Delaunay polytope P. When we move d_P inside F_P, the Delaunay polytope P changes, while its affine type remain the same. In other words, like the rank of P, the affine type of P is an invariant of the face F_P.

The above movement of d_P inside F_P corresponds to a perturbation of each basis of L, and, therefore, of each Gram matrix (i.e., each quadratic form) related to L. In this paper, we show that there is a one-to-one correspondence between the space spanned by F_P and the space $\mathcal{B}(P)$ spanned by the set of perturbed quadratic forms. Hence, those two spaces have the same

dimension. It is shown here, that if one knows the coordinates of vertices of P in a basis, then it is simpler to compute $\dim(\mathcal{B}(P))$ than $\dim(F_P)$. This fact is illustrated by computations of ranks of cross polytope and half-cube.

In the last section, we describe a non-basic repartitioning Delaunay polytope recently discovered by the first author.

2 Equalities of negative type and hypermetric

A sphere $S = S(c, r)$ of radius r and center c in an n-dimensional lattice L is said to be an *empty sphere* if the following two conditions hold:

(i) $\|a - c\|^2 \geq r^2$ for all $a \in L$,

(ii) the set $S \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$ with $S(c, r)$ an empty sphere.

Denote by $L(P)$ the lattice generated by P. In this paper, we can suppose that P is *generating* in L, i.e., that $L = L(P)$. A subset $V \subseteq V(P)$ is said to be $\mathbb{K}$-*generating*, with $\mathbb{K}$ being a ring, if every vertex $w \in V(P)$ has a representation $w = \sum_{v \in V} z(v)v$ with $1 = \sum_{v \in V} z(v)$ and $z(v) \in \mathbb{K}$. If $|V| = n + 1$, then V is called an $\mathbb{K}$-*affine basis*; the Delaunay polytope P is called $\mathbb{K}$-*basic* if it admits at least one $\mathbb{K}$-affine basis. In this work $\mathbb{K}$ will be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and if the ring is not precised, it is $\mathbb{Z}$. Furthermore, let

$$Y(P) = \{y \in \mathbb{Z}^{V(P)} : \sum_{v \in V(P)} y(v)v = 0, \quad \sum_{v \in V(P)} y(v) = 0\} \tag{2}$$

be the $\mathbb{Z}$-module of all integral dependencies on $V(P)$. If the Delaunay polytope P is a simplex, then $Y(P) = \{0\}$.

A dependency on $V(P)$ implies some dependencies between distances $d_P(u, v)$ as follows. Let c be the center of the empty sphere S circumscribing P. Then all vectors $v - c$, $v \in V(P)$, have the same norm $\|v - c\|^2 = r^2$, where r is the radius of the sphere S. Hence,

$$d_P(u, v) = \|u - v\|^2 = \|u - c - (v - c)\|^2 = 2(r^2 - \langle u - c, v - c\rangle). \tag{3}$$

Multiplying this equality by $y(v)$ and summing over $v \in V(P)$, we get

$$\sum_{v \in V(P)} y(v)d_P(u, v) = 2r^2 \sum_{v \in V(P)} y(v) - 2\langle u - c, \sum_{v \in V(P)} y(v)(v - c)\rangle.$$

Since $y \in Y(P)$, we obtain the following important equality

$$\sum_{v \in V(P)} y(v)d_P(u, v) = 0, \text{ for any } u \in V(P) \text{ and } y \in Y(P). \tag{4}$$

Denote by $\mathcal{S}_{dist}(P)$ the system of equations (4) for all integral dependencies $y \in Y(P)$ and all $u \in V(P)$, where the distances $d_P(u,v)$ are considered as unknowns.

Multiplying the equality (4) by $y(u)$ and summing over all $u \in V(P)$, we obtain

$$\sum_{u,v \in V(P)} y(u)y(v)d_P(u,v) = 0. \tag{5}$$

This equality is called an equality of *negative type* and the system of such equality is denoted $\mathcal{S}_{neg}(P)$. Hence, the equalities of $\mathcal{S}_{neg}(P)$ are implied by the one of $\mathcal{S}_{dist}(P)$.

Each integral dependency $y \in Y(P)$ determines the following representation of a vertex $w \in V(P)$ as an integer combination of vertices from $V(P)$:

$$w = w + \sum_{v \in V(P)} y(v)v = \sum_{v \in V(P)} y^w(v)v,$$

where

$$y^w(v) = \begin{cases} y(v) & \text{if } v \neq w, \\ y(w)+1 & \text{if } v = w \end{cases} \quad \text{and} \quad \sum_{v \in V(P)} y^w(v) = 1.$$

Let δ_w be the indicator function of $V(P)$: $\delta_w(v) = 0$ if $v \neq w$, and $\delta_w(w) = 1$. Obviously, δ_w is y^w for the trivial representation $w = w$. We have $y^w = y + \delta_w$. Conversely, every representation $w = \sum_{v \in V(P)} y^w(v)v$ provides the dependency $y = y^w - \delta_w \in Y(P)$. Substituting $y = y^w - \delta_w$ in (5), we obtain the following equality

$$\sum_{u,v \in V(P)} y(u)y(v)d_P(u,v) = \sum_{u,v \in V(P)} y^w(u)y^w(v)d_P(u,v) - 2\sum_{v \in V(P)} y^w(v)d_P(w,v).$$

Since $d_P(w,w) = 0$, we can set $y^w = y$ in the last sum. For any $w \in V(P)$, we use this equality in the following form using equations (4) and (5)

$$\sum_{u,v \in V(P)} y^w(u)y^w(v)d_P(u,v) = \sum_{u,v \in V(P)} y(u)y(v)d_P(u,v) + 2\sum_{v \in V(P)} y(v)d_P(w,v) = 0. \tag{6}$$

The equality

$$\sum_{u,v \in V(P)} z(u)z(v)d_P(u,v) = 0, \text{ where } \sum_{v \in V(P)} z(v) = 1, \ z(v) \in \mathbb{Z},$$

is the hypermetric equality. Denote by $\mathcal{S}_{hyp}(P)$ the system of all hypermetric equalities which hold for $d_P(u,v)$, considering the distances $d_P(u,v)$ as unknowns.

In [3], the following lemma is proved. For the sake of completeness, we give its short proof.

Lemma 1 *Let P be a Delaunay polytope with vertex-set $V(P)$. Let $y^w \in \mathbb{Z}^{V(P)}$, such that $\sum_{v \in V(P)} y^w(v) = 1$. Then the following assertions are equivalent*

(i) a vertex $w \in V(P)$ has the representation $w = \sum_{v \in V(P)} y^w(v)v$;

(ii) the distance d_P satisfies the hypermetric equality $\sum_{u,v\in V(P)} y^w(u)y^w(v)d_P(u,v)=0$.

Proof. (i)⇒(ii) Obviously, $y=y^w-\delta_w$, is a dependency, i.e. $y\in Y(P)$. Hence, this implication follows from the equalities (6), (4) and (5).

(ii)⇒(i) Substituting the expression (3) for d_P in the hypermetric equality of (ii) we obtain the equality

$$2r^2-2\|\sum_{v\in V(P)} y^w(v)(v-c)\|^2=0.$$

Obviously, $\sum_{v\in V(P)} y^w(v)c=c$ and $\sum_{v\in V(P)} y^w(v)v$ is a point of $L(P)$. Denote this point by w. Then the above equality takes the form $\|w-c\|^2=r^2$. Hence, w lies on the empty sphere circumscribing P. Therefore, $w\in V(P)$ and (i) follows. □

According to Lemma 1, each hypermetric equality of the system $\mathcal{S}_{hyp}(P)$ corresponds to a representation y^w of a vertex $w\in V(P)$. Since the relation $y=y^w-\delta_w$ gives a one-to-one correspondence between dependencies on $V(P)$ and non-trivial representations y^w of vertices $w\in V(P)$, we can prove the following assertion:

Lemma 2 *The systems of equations $\mathcal{S}_{dist}(P)$ and $\mathcal{S}_{hyp}(P)$ are equivalent, i.e., their solution sets coincide.*

Proof. The equality (6) shows that each equation of the system $\mathcal{S}_{hyp}(P)$ is implied by equations of the system $\mathcal{S}_{dist}(P)$.

Now, we show the converse implication. Suppose the unknowns $d(u,v)$ satisfy all hypermetric equalities of the system $\mathcal{S}_{hyp}(P)$. The equality (6) implies the equality

$$2\sum_{v\in V(P)} y(v)d(w,v)=-\sum_{u,v\in V(P)} y(u)y(v)d(u,v),$$

where $y=y^w-\delta_w$. This shows that, for the dependency y on $V(P)$, $\sum_{v\in V(P)} y(v)d(w,v)$ does not depend on w; denote it by $A(y)$. Hence, we have

$$-2A(y)=\sum_{u,v\in V(P)} y(u)y(v)d(u,v)=A(y)\sum_{u\in V(P)} y(u).$$

According to equation (2), the last sum equals zero. This implies the equalities (5) and hence the equalities of the system $\mathcal{S}_{dist}(P)$. □

Obviously, the space determined by the system $\mathcal{S}_{hyp}(P)$ (and also of the system $\mathcal{S}_{dist}(P)$) is a subspace $X(P)$ of the space spanned by all distances $d(u,v)$, $u,v\in V(P)$. The dimension of $X(P)$ is the rank of P. According to Lemma 2, in order to compute the rank of P, we can use only equations of the system $\mathcal{S}_{dist}(P)$.

Let $V_0=\{v_0,v_1,\dots,v_n\}$ be an $\mathbb{R}$-affine basis of P. Then each vertex $w\in V(P)$ has a unique representation through vertices of V_0 as follows

$$w=\sum_{v\in V_0} x(v)v,\ \ \sum_{v\in V_0} x(v)=1,\ x(v)\in\mathbb{R}.$$

Since the vertices of P are points of a lattice, in fact, $x(v) \in \mathbb{Q}$. Hence, the above equation can be rewritten as an integer dependency

$$y_w(w)w + \sum_{v \in V_0} y_w(v)v = 0, \ y_w(w) + \sum_{v \in V_0} y_w(v) = 0, \text{ with } y_w(v) \in \mathbb{Z}.$$

One sets $y_w(u) = 0$ for $u \in V(P) - (V_0 \cup \{w\})$ and gets $y_w \in Y(P)$. Any dependency $y \in Y(P)$ is a rational combination of dependencies y_w, $w \in V(P) - V_0$. Hence, the following equality holds:

$$\beta y = \sum_{w \in V(P) - V_0} \beta_w y_w, \text{ with } \beta_w \in \mathbb{Z} \text{ and } 0 < \beta \in \mathbb{Z}$$

Since the equalities (4) are linear over $y \in Y(P)$, the dependencies y_w, $w \in V(P) - V_0$ provide the following system, which is equivalent to $\mathcal{S}_{dist}(P)$

$$y_w(w)d_P(u, w) + \sum_{v \in V_0} y_w(v)d_P(u, v) = 0, \text{ with } u \in V(P) \text{ and } w \in V(P) - V_0. \tag{7}$$

We see that, for $u \in V(P) - V_0$, the distance $d_P(u, w)$, $w \in V(P) - V_0$, is also expressed through distances between u and $v \in V_0$. But for $u \in V_0$, the distance $d_P(u, w)$ is expressed through distances between $u, v \in V_0$. This implies that the distance $d_P(u, w)$ for $u, w \in V(P) - V_0$ can be also represented through distances $d_P(u, v)$ for $u, v \in V_0$. Hence, the dimension of $X(P)$ does not exceed $\frac{n(n+1)}{2}$, where $n + 1 = |V_0|$, which is the dimension of the space of distances between the vertices of V_0.

In order to obtain dependencies between $d_P(u, v)$ for $u, v \in V_0$, we use equation (7) for $u = w$. Since $d_P(w, w) = 0$, we obtain the equations

$$\sum_{v \in V_0} y_w(v)d_P(v, w) = 0, \ w \in V(P) - V_0.$$

Multiplying the above equation by $y_w(w)$ and using equation (7), we obtain

$$0 = \sum_{u \in V_0} y_w(u)(y_w(w)d_P(u, w)) = -\sum_{u \in V_0} y_w(u) \sum_{v \in V_0} y_w(v)d_P(u, v).$$

So, we obtain the following main equations for dependencies between $d_P(u, v)$ for $u, v \in V_0$

$$\sum_{u,v \in V_0} y_w(u)y_w(v)d_P(u, v) = 0, \ w \in V(P) - V_0. \tag{8}$$

Note, that if V_0 is an affine basis of $L(P)$, then one can set $y_w(w) = -1$. In this case, the equation $y_w(w) + \sum_{v \in V_0} y_w(v) = 0$ takes the form $\sum_{v \in V_0} y_w(v) = 1$. This implies that the above equations are hypermetric equalities for a $\mathbb{Z}$-basic Delaunay polytope P. If P is $\mathbb{Z}$-basic, then the distance d_P restricted to the set V_0 lies on the face of the cone $HYP(V_0)$ determined by the hypermetric equalities (8). But if P is not $\mathbb{Z}$-basic, then the equations (8) are not hypermetric, and the distance d_P restricted on the set V_0 lies inside the cone $HYP(V_0)$. On the other hand,

the distance d_P on the whole set $V(P)$ lies on the boundary of the cone $HYP(V(P))$. This implies that, in this case, the rank of d_P restricted to V_0 is greater than the rank of d_P on $V(P)$.

This can be explained as follows. We can consider the cone $HYP(V_0)$ as a projection of $HYP(V(P))$ on a face of the positive orthant $\mathbb{R}^N_+$, where $N = |V(P)|$. This face is determined by the equations $d(u,v) = 0$ for $v \in V(P) - V_0$ or/and $u \in V(P) - V_0$. By this projection, the distance d_P, lying on the boundary of the cone $HYP(V(P))$, is projected into the interior of the cone $HYP(V_0)$. This hypermetric space corresponds to a wall of an L-type domain, which lies inside the cone $HYP(V_0)$.

But, in order to compute the rank of P, it is sufficient to find the dimension of the space determined by the system (8).

3 Dependencies between lattice vectors

Now we go from affine realizations to linear realizations. Take $v_0 \in V_0$ as origin of the lattice $L(P)$ and choose the lattice vectors $a_i = a(v_i) = v_i - v_0$, $1 \le i \le n$ such that $\{a_i : 1 \le i \le n\}$ forms a $\mathbb{Q}$-basis of $L(P)$. If P is basic, we can choose v_i such that $\{a_i : 1 \le i \le n\}$ is a $\mathbb{Z}$-basis of $L(P)$. Using the expressions $d_P(v_i, v_j) = \|a_i - a_j\|^2$, it is easy to verify that there is the following relation between distances $d_P(u,v)$, $u, v \in V_0$, and inner products $\langle a_i, a_j\rangle$:

$$d_P(v_i, v_0) = \|a_i\|^2,\ d_P(v_i, v_j) = \|a_i\|^2 - 2\langle a_i, a_j\rangle + \|a_j\|^2.$$

And conversely,

$$\|a_i\|^2 = d_P(v_i, v_0),\ \langle a_i, a_j\rangle = \frac{1}{2}(d_P(v_i, v_0) + d_P(v_j, v_0) - d_P(v_i, v_j)).$$

This shows that there is a one-to-one correspondence between the set of distances $d_P(v_i, v_j)$, $0 \le i < j \le n$, and the set of inner products $\langle a_i, a_j\rangle$, $1 \le i \le j \le n$.

We substitute the above expressions for $d_P(v_i, v_j)$, $0 \le i, j \le n$, into the equations (8), where we set $y_w(i) = y_w(v_i)$, and use the equality $\sum_{i=0}^n y_w(i) = -y_w(w)$. We obtain the following important equations

$$-y_w(w) \sum_{i=1}^n y_w(i)\|a_i\|^2 = \|\sum_{i=1}^n y_w(i) a_i\|^2,\ w \in V(P) - V_0. \tag{9}$$

We can obtain the equation (9) directly, as follows. For $v \in V(P)$, the vector $a(v) = v - v_0$ is a lattice vector of $L(P)$. For $y \in Y(P)$, we have obviously $\sum_{v \in V(P)} y(v)a(v) = 0$. In particular, for $y = y_w$, this equation has the form

$$y_w(w)a(w) + \sum_{i=1}^n y_w(i)a_i = 0$$

and allows to represent the vectors $a(w)$ in the $\mathbb{Q}$-basis $\{a_i : 1 \le i \le n\}$.

Recall that the lattice vector $a(w)$ of each vertex $w \in V(P)$ of a Delaunay polytope P satisfies the equation $\|a(w) - c\|^2 = r^2$. Since $v_0 \in V$, $a(v_0) = 0$, which implies $\|c\|^2 =$

$\|0-c\|^2 = r^2$. The vertex-set of P provides the following system of equations $\|a(w)-c\|^2 = \|c\|^2$, $w \in V(P)$, i.e.,

$$2\langle c, a(w)\rangle = \|a(w)\|^2, \ w \in V(P). \tag{10}$$

Since $y_w(w)a(w) = -\sum_{i=1}^n y_w(i)a_i$, the above equations take the form

$$-y_w(w)\sum_{i=1}^n y_w(i)2\langle c, a_i\rangle = \|\sum_{i=1}^n y_w(i)a_i\|^2. \tag{11}$$

Recall that $a_i = a(v_i)$. Hence, the vertices v_i give $2\langle c, a_i\rangle = \|a_i\|^2$, and the above equation takes the form of equation (9).

We will use the equations (9) mainly for basic Delaunay polytopes. In this case, we can set $y_w(w) = -1$, and $a_i = b_i$, $1 \le i \le n$, where $B = \{b_i : 1 \le i \le n\}$ is the basis of $L(P)$ consisting of lattice vectors of the basic Delaunay polytope P.

For a given $\mathbb{Q}$-affine basis $V_0 \subseteq V(P)$ of a Delaunay polytope P, the set of affine dependencies $\{y_w \in Y(P) : w \in V(P) - V_0\}$ is uniquely determined up to integral multipliers and form a $\mathbb{Q}$-basis of the $\mathbb{Z}$-module $Y(P)$. This implies that the equations (9) determine a subspace

$$\mathcal{A}(P) = \{a_{ij} : -y_w(w)\sum_{i=1}^n y_w(i)a_{ii} = \sum_{1\le i,j\le n} y_w(i)y_w(j)a_{ij}, \ y_w \in Y(P), w \in V(P) - V_0\}$$

in the $\frac{n(n+1)}{2}$-dimensional space of all symmetric $n \times n$-matrices $a_{ij} = a_{ji}$, $1 \le i < j \le n$,

Since there is a one-to-one correspondence between distances $d(v_i, v_j)$, $0 \le i < j \le n$, and inner products $a_{ij} = \langle a_i, a_j\rangle$, $1 \le i \le j \le n$, the dimension of the subspace $\mathcal{A}(P)$ is equal to the rank of P. So, in order to compute the rank of P, we have to find the dimension of $\mathcal{A}(P)$.

4 The space $\mathcal{B}(P)$ and our computational method

Fix a basis $B = \{b_i : 1 \le i \le n\}$ of the lattice L. Every lattice vector $a(v)$, $v \in V(P)$, has a unique representation $a(v) = \sum_{i=1}^n z_i(v)b_i$. Define $\mathcal{Z}_B(P) = \{z_i(v) : 1 \le i \le n, v \in V(P)\}$.

Recall that the cone $\mathcal{P}_n$ of all positive semi-definite forms on n variables is partitioned into L-type domains. Each L-type domain is an open polyhedral cone of dimension k, where $1 \le k \le \frac{n(n+1)}{2}$. It consists of form having affinely equivalent partitions into Delaunay polytopes, i.e. *Delaunay partitions*. More exactly, an L-type domain is the set of quadratic forms $f(x) = \|\sum_{i=1}^n x_ib_i\|^2$ having the same set of matrices $\mathcal{Z}_B(P)$ for all non-isomorphic Delaunay polytopes P of its Delaunay partition. So, this set is not changed when the basis B changes such that the form $f(x)$ belongs to the same L-type domain. In other words, $\mathcal{Z}_B(P)$ is an invariant of this L-type domain.

We set $z_{ij} = z_i(v_j)$ for $v_j \in V_0 - \{v_0\}$, $1 \le j \le n$. The matrix $Z_B = (z_{ij})_1^n$ is non-degenerate and gives a correspondence between the linear bases of P and bases of $L(P)$. In particular, this correspondence maps the space $\mathcal{A}(P)$ in the space $\mathcal{B}(P)$ of matrices $b_{ij} = \langle b_i, b_j\rangle$ of the quadratic form $f(x)$. If P is basic and $b_i = a_i$, $1 \le i \le n$, then Z_B is the identity matrix I, and $\mathcal{A}(P) = \mathcal{B}(P)$.

Substituting in the equations (10) the above representations of the vectors $a(v)$, $v \in V(P)$, in the basis B, we obtain explicit equations, determining the space $\mathcal{B}(P)$. In fact, we have

$$2\sum_{i=1}^{n} z_i(v)\langle c, b_i\rangle = \sum_{1\le i,j\le n} z_i(v)z_j(v)b_{ij}, \ v \in V(P). \tag{12}$$

We have the following $\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$ parameters in the equations (12):

$$b_{ij} = \langle b_i, b_j\rangle, \ 1 \le i \le j \le n, \text{ and } \langle c, b_i\rangle, \ 1 \le i \le n,$$

Hence, all these parameters can be represented through a number of independent parameters. This number is just the rank of P. Recall that a Delaunay polytope is called *extreme* if $\mathrm{rk}(P) = 1$. Hence, in order to be extreme, a Delaunay polytope should have at least $\frac{n(n+3)}{2}$ vertices.

Note that, for $v = v_0$, the equation (12) is an identity, since $a(v_0) = 0$ and therefore $z_i(v_0) = 0$ for all i. So, we have $|V(P)| - 1$ equations (12). For $v = v_i$, $1 \le i \le n$, one gets n equations that give a representation of the parameters $\langle c, b_i\rangle$, $1 \le i \le n$ in terms of the parameters $\langle b_i, b_j\rangle$, $1 \le i \le j \le n$. Hence, the equations (12), for $v \in V(P) - V_0$, allow to find dependencies between the main parameters $\langle b_i, b_j\rangle$. Now, we write out explicitly dependencies between $\langle b_i, b_j\rangle$.

Since the basic vectors $b_i \in B$ are mutually independent, a dependency $\sum_{v\in V} y(v)a(v) = 0$ implies the dependencies $\sum_{v\in V} y(v)z_i(v) = 0$ between the coordinates $z_i(v)$, $1 \le i \le n$.

Multiplying equation (12) by $y(v)$, and summing over all $v \in V(P)$, we obtain that the $\mathbb{Z}$-module $Y(P)$ determines the following subspace of the space of parameters $b_{ij} = \langle b_i, b_j\rangle$:

$$\mathcal{B}(P) = \{b_{ij} : \sum_{i,j=1}^{n}(\sum_{v\in V} y(v)z_i(v)z_j(v))b_{ij} = 0, \ y \in Y(P)\}.$$

In the Delaunay partition of the lattice $L(P)$, there are infinitely many Delaunay polytopes equivalent to P. Each of them has the form $a \pm P$, where $a = \sum_{i=1}^{n} z_i^a b_i$ is an arbitrary lattice vector of $L(P)$. Now, we show that the space $\mathcal{B}(P)$ is independent on a representative of P in $L(P)$, i.e., that $\mathcal{B}(P) = \mathcal{B}(a \pm P)$.

Let $v_a = a \pm v$ be the vertex of the polytope $a \pm P$ corresponding to a vertex v of P. Obviously, $z_i(v_a) = z_i^a \pm z_i(v)$. Substituting these values of $z_i(v_a)$ into the equations determining $\mathcal{B}(a \pm P)$, we obtain

$$\sum_{v_a} y(v_a)z_i(v_a)z_j(v_a) = \sum_{v\in V(P)} y(v)(z_i^a z_j^a \pm z_i^a z_j(v) + z_i(v)z_j(v)).$$

Since y is a dependency between vertices of P, the sums with z_1^a equal zero. This shows that $\mathcal{B}(P)$ does not depend on a representative of P.

Since the equalities determining the space $\mathcal{B}(P)$ are linear in y, we can consider these equalities only for basic dependencies y_w, $w \in V(P) - V_0$. We obtain the following main system of equations describing dependencies between the parameters b_{ij};

$$\sum_{i,j=1}^{n}(\sum_{v\in V} y_w(v)z_i(v)z_j(v))b_{ij} = 0, \ w \in V(P) - V_0. \tag{13}$$

A unimodular transformation maps a basis of $L(P)$ into another basis. This transformation generates a transformation which maps the space $\mathcal{B}(P)$ into another space related to P. The dimension of the space $\mathcal{B}(P)$ is an invariant of the lattice $L(P)$ generated by P.

In [1], a *non-rigidity degree* of a lattice was defined. In terms of this paper, the non-rigidity degree of a lattice L is the dimension of the intersection of spaces $\mathcal{B}(P)$ related to all non-isomorphic Delaunay polytopes of a star of Delaunay polytopes of L. Hence,

$$\mathrm{nrd}(L) = \dim(\cap_P \mathcal{B}(P)).$$

In fact, the space $\cap_P \mathcal{B}(P)$ is the supporting space of the L-type domain of the lattice L.

5 Centrally symmetric construction

In many cases, the computation of the rank of a Delaunay polytope P using the equations (12) is easier than by using the hypermetric equalities generated by P. We demonstrate this by giving a simpler proof of Lemma 15.3.7 of [3]. Recall that a Delaunay polytope is either centrally symmetric or asymmetric. Let c be the center of the empty sphere circumscribing P. For any $v \in V(P)$, the point $v^* = 2c - v$ is centrally symmetric to v. If P is centrally symmetric, then $v^* \in V(P)$ for all $v \in V(P)$. If P is asymmetric, then $v^* \notin V(P)$ for all $v \in V(P)$.

Lemma 3 *Let P be an n-dimensional basic centrally symmetric Delaunay polytope of a lattice L with the following properties:*

1. *The origin $0 \in V(P)$ and the vectors e_i, $1 \le i \le n$, are basic vectors of L, whose endpoints are vertices of P.*

2. *The intersection $P_1 = P \cap H$ of P with the hyperplane H generated by the vectors e_i, $1 \le i \le n-1$, is an asymmetric Delaunay polytope of the lattice $L_1 = L \cap H$.*

3. *If the endpoint v_n of the basic vector e_n is v^* for some $v \in V(P_1)$, then there is a vertex $u \in V(P)$ such that $u \neq v, v^*$ for all $v \in V(P_1)$.*

Then $\mathrm{rk}(P) \le \mathrm{rk}(P_1)$.

Proof. It is sufficient to prove that the n parameters $\langle e_i, e_n \rangle$, $1 \le i \le n$, can be expressed through the parameters $a_{ij} = \langle e_i, e_j \rangle$, $1 \le i \le j \le n-1$.

Let c be the center of P. Obviously, $2c = 0^* \in V(P)$. Since P_1 is asymmetric, $2c \notin L_1$. It is easy to see that $2c = a_0 + z e_n$, with $a_0 = \sum_{i=1}^{n-1} y_i e_i \in L_1$ and $0 \neq z \in \mathbb{Z}$. Hence, the equation $2\langle c, e_i \rangle = \|e_i\|^2$ takes the form $\langle a_0 + z e_n, e_i \rangle = \|e_i\|^2$, and the parameters $\langle e_i, e_n \rangle$ are represented through the parameters $\langle e_i, e_j \rangle$ as follows

$$\langle e_i, e_n \rangle = \frac{1}{z}(\|e_i\|^2 - \langle a_0, e_i \rangle), \ 1 \le i \le n-1.$$

Now, using the equation $2\langle c, e_n \rangle = \|e_n\|^2$, we obtain $\langle a_0 + z e_n, e_n \rangle = \|e_n\|^2$, i.e., $\|e_n\|^2(1 - z) = \langle a_0, e_n \rangle = \sum_{i=1}^{n-1} y_i \langle e_i, e_n \rangle$. Hence, if $z \neq 1$, we can represent $\|e_n\|^2$ through $\langle e_i, e_j \rangle$,

$1 \le i \le j \le n-1$, too. But if $z = 1$, then the endpoint v_n of e_n belongs to $(V(P_1))^*$. In this case there is a vertex u such that $u = \sum_{i=1}^n z_i e_i = u_0 + z_n e_n$, where $u_0 \in L_1$ and $z_n \neq 0, 1$. Using the equation $2\langle c, u\rangle = \|u\|^2$, where now $2c = a_0 + e_n$, we have $\langle a_0 + e_n, u_0 + z_n e_n\rangle = \|u_0 + z_n e_n\|^2$. This equation gives

$$\|e_n\|^2 = \frac{1}{z_n(z_n-1)}[\langle a_0 - u_0, u_0\rangle + \langle z_n a_0 + (1-2z_n)u_0, e_n\rangle].$$

The strict inequality $\mathrm{rk}(P) < \mathrm{rk}(P_1)$ is possible if some vertices of the set $V(P) - V(P_1)$ provide additional relations between the parameters $\langle e_i, e_j\rangle$, $1 \le i \le j \le n-1$. □

Examples, where $\mathrm{rk}(P) < \mathrm{rk}(P_1)$, can be given by some extreme Delaunay polytopes.

Corollary 1 *Let P be a basic centrally symmetric Delaunay polytope satisfying the conditions of Lemma 3. P is extreme if P_1 is extreme.*

6 Computing the rank of simplexes, cross-polytopes and half-cubes

Simplices. Let Σ be an n-dimensional simplex with vertices 0, v_i, $1 \le i \le n$. The vertex v_i is the end-point of the basic vector e_i, $1 \le i \le n$. We have only n equations $2\langle c, e_i\rangle = \|e_i\|^2$ determining only the coordinates of the center c of Σ in the basis $\{e_i : 1 \le i \le n\}$. Since there is no relation between the $\frac{n(n+1)}{2}$ parameters $\langle e_i, e_j\rangle = a_{ij}$, all these parameters are independent. Hence,

$$\dim(\mathcal{B}(\Sigma)) = \frac{n(n+1)}{2}, \text{ i.e., } \mathrm{rk}(\Sigma) = \frac{n(n+1)}{2}.$$

Cross-polytopes. An n-dimensional cross-polytope β_n is a basic centrally symmetric Delaunay polytope. It is the convex hull of $2n$ end-points of n linearly independent segments intersecting in the center c of the circumscribing sphere. The set $V(\beta_n)$ is partitioned into two mutually centrally symmetric n-subsets each of which is the vertex-set of an $(n-1)$-dimensional simplex Σ. So, $V(\beta_n) = V(\Sigma) \cup V(\Sigma^*)$. Let $V(\Sigma) = \{0, v_i : 1 \le i \le n-1\}$. All $\mathbb{Z}$-affine bases of β_n are of the same type: $n-1$ basic vectors e_i, $1 \le i \le n-1$, with end vertices v_i, are basic vectors of the simplex Σ, and $e_n = 2c$, which is the segment which connects the vertex 0 with its opposite vertex 0^*. Let a_i be the lattice vector endpoint of which is the vertex $v_i^* \in \Sigma^*$. Obviously, $a_i = 2c - e_i = e_n - e_i$. The equality $2\langle c, a_i\rangle = \|a_i\|^2$ gives $\langle e_i, e_n\rangle = \|e_i\|^2$, $1 \le i \le n-1$. So, we obtain $n-1$ independent relations between the parameters $\langle e_i, e_j\rangle$, and they are the only relations. Hence,

$$\mathrm{rk}(\beta_n) = \frac{n(n+1)}{2} - (n-1).$$

(Cf., the first formula on p.232 of [3].)

Half-cubes. Take $N = \{1, 2, \ldots, n\}$, a basis $(e_i)_{i\in N}$ and defines $e(T) = \sum_{i\in T} e_i$ for any $T \subseteq N$. Call a set $T \subseteq N$ *even* if its cardinality $|T|$ is even. A half-cube $h\gamma_n$ is the convex hull

of endpoints of all vectors $e(T)$ for all even $T \subseteq N$. Note that $h\gamma_3$ is a simplex, and $h\gamma_4$ is the cross-polytope β_4. Hence,

$$\mathrm{rk}(h\gamma_3) = \frac{3(3+1)}{2} = 6, \text{ and } \mathrm{rk}(h\gamma_4) = \frac{4(4+1)}{2} - 3 = 7.$$

The rank of $h\gamma_n$ is computed from the following system of equations:

$$2\langle c, e(T)\rangle = \|e(T)\|^2, \ T \subseteq N, \ T \text{ is even.} \tag{14}$$

Let T_1 and T_2 be two disjoint even subsets of N. Since the set $T = T_1 \cup T_2$ is even, we have

$$2\langle c, e(T_1 \cup T_2)\rangle = 2\langle c, e(T_1) + e(T_2)\rangle = \|e(T_1) + e(T_2)\|^2 = \|e(T_1)\|^2 + \|e(T_2)\|^2 + 2\langle e(T_1), e(T_2)\rangle.$$

Comparing this equation with the equations (14) for $T = T_1$ and $T = T_2$, we obtain that for any two disjoint even subsets the following *orthogonality conditions* hold:

$$\langle e(T_1), e(T_2)\rangle = 0, \text{ if } T_1 \cap T_2 = \emptyset, \ T_i \subset N, \text{ and } T_i \text{ is even}, \ \ i = 1, 2.$$

Note that, for $n = 3$, we have no orthogonality condition. If $n \geq 4$, take 4 elements i, j, k and l and write three equalities corresponding to three partitions:

$$\langle e_i + e_j, e_k + e_l\rangle = 0, \ \langle e_i + e_k, e_j + e_l\rangle = 0, \ \langle e_i + e_l, e_j + e_k\rangle = 0.$$

It is easy to verify that these equalities are equivalent to the following three equalities

$$\langle e_i, e_j\rangle + \langle e_k, e_l\rangle = 0, \ \langle e_i, e_k\rangle + \langle e_j, e_l\rangle = 0, \ \langle e_i, e_l\rangle + \langle e_j, e_k\rangle = 0. \tag{15}$$

In the particular case $n = 4$, we conclude again that $\mathrm{rk}(h\gamma_4) = \frac{4(4+1)}{2} - 3 = 7$.

We show that, for $n \geq 5$, the orthogonality conditions are equivalent to mutual orthogonality of all vectors e_i, $i \in N$. To this end, it is sufficient to consider even subsets of cardinality two and use equation (15) for each quadruple $\{i, j, k, l\} \subseteq N$. Considering arbitrary subsets of N of cardinality 4, we obtain that, for $n \geq 5$, the system of equalities (15) for all quadruples has the following unique solution

$$\langle e_i, e_j\rangle = 0, \ 1 \leq i < j \leq n, \text{ for } n \geq 5.$$

So, all the basic vectors are mutually orthogonal. Obviously, the orthogonality of basic vectors implies the orthogonality conditions. Hence, the only independent parameters are the n parameters $\|e_i\|^2$, $i \in N$. This implies that

$$\mathrm{rk}(h\gamma_n) = n \text{ if } n \geq 5.$$

Note that we use a basis of $h\gamma_n$, which is not a basis of the lattice generated by $h\gamma_n$. But the spaces $\mathcal{B}(P)$ have the same dimension for all bases. See another proof in [4].

7 A non-basic repartitioning Delaunay polytope

The example P_0 given in this section is 12 dimensional; its 14 vertices belong to two disjoint sets of vertices of regular simplexes Σ_i^2, $i = 1, 2$, of dimension 2, and two disjoint sets of vertices of regular simplexes Σ_i^3, $i = 1, 2$, of dimension 3.

Let $V(\Sigma_i^q)$ be the vertex-set of the four simplex Σ_i^q, $i = 1, 2$, $q = 2, 3$. Then $V = \cup V(\Sigma_i^q)$ is the vertex-set of P_0. The distances between the vertices of P_0 are as follows

$$d(u,v) = \begin{cases} 7 & \text{if} \quad u, v \in \Sigma_i^q, & i = 1, 2, & q = 2, 3; \\ 6 & \text{if} \quad u \in \Sigma_i^2, & v \in \Sigma_i^3, & i = 1, 2; \\ 10 & \text{if} \quad u \in \Sigma_1^2, & v \in \Sigma_2^2; & \\ 12 & \text{if} \quad u \in \Sigma_1^3, v \in \Sigma_2^3 \text{ or} & u \in \Sigma_1^2, v \in \Sigma_2^3, \text{ or} & u \in \Sigma_2^2, v \in \Sigma_1^3. \end{cases}$$

We show that, for every $u \in V$, the set $V - \{u\}$ is an $\mathbb{R}$-affine basis of P_0. In fact, let $V - \{u\} = \{v_i : 0 \leq i \leq 12\}$ and let $a_i = v_i - v_0$, $1 \leq i \leq 12$. For the Gram matrix $a_{ij} = \langle a_i, a_j \rangle$, we have $a_{ii} = \|a_i\|^2 = \|v_i - v_0\|^2 = d(v_i, v_0)$. The relations between a_{ij} and $d(v_i, v_j)$ are $a_{ij} = \frac{1}{2}(d(v_i, v_0) + d(v_j, v_0) - d(v_i, v_j))$. Now, one can verify that the Gram matrix (a_{ij}) is not singular. Hence, $\{a_i : 1 \leq i \leq n\}$ is a basis, i.e. the dimension of P_0 is, in fact, 12.

The space $Y(P_0)$ of affine dependencies on vertices of P_0 is one-dimensional. For $v \in V$, let

$$y(v) = \begin{cases} 3 & \text{if} \quad v \in \Sigma_1^2, \\ -3 & \text{if} \quad v \in \Sigma_2^2, \\ 2 & \text{if} \quad v \in \Sigma_2^3, \\ -2 & \text{if} \quad v \in \Sigma_1^3. \end{cases}$$

Obviously, $\sum_{v \in V} y(v) = 0$. It is easy to verify that for any $u \in V$ the following equality holds

$$\sum_{v \in V} y(v) d(u,v) = 0. \tag{16}$$

Let $S(c,r)$ be the sphere circumscribing P_0. Then $\|v - c\|^2 = r^2$ for all $v \in V$. We have $d(u,v) = \|u - v\|^2 = \|(u - c) - (v - c)\|^2 = 2(r^2 - \langle u - c, v - c \rangle)$. Since $\sum_{v \in V} y(v) = 0$, the equality (16) takes the form

$$\langle u - c, \sum_{v \in V} y(v)(v - c) \rangle = 0, \text{ i.e., } \langle u - c, \sum_{v \in V} y(v)(v - c) \rangle = \langle u - c, \sum_{v \in V} y(v) v \rangle = 0.$$

Since this equality holds for all $u \in V$, and $\{u - c \mid u \in V\}$ span $\mathbb{R}^{12}$, we obtain $\sum_{v \in V} y(v) v = 0$, i.e., $y \in Y(P_0)$. Since $Y(P_0)$ is one dimensional and the coefficient of y have greatest common divisor 1, one has $Y(P_0) = y\mathbb{Z}$.

Using the basis $\{a_i : 1 \leq i \leq 12\}$, for non-basic $a(w)$, we obtain $a(w) = -\frac{1}{y(w)} \sum_{i=1}^{12} y(v_i) a_i$. Since there exist a i such that $\frac{y(v_i)}{y(w)} \notin \mathbb{Z}$ for any choice of $w \in V$, the polytope P_0 is not basic and the $\mathbb{Q}$-basis $\{a_i : 1 \leq i \leq n\}$ is not a $\mathbb{Z}$-basis of any lattice L having P_0 as a Delaunay polytope.

Remarking that we can put the vector y in equation (9), we obtain the following equation

$$-y(w)\sum_{i=1}^{12} y(v_i)\|a_i\|^2 = \|\sum_{i=1}^{12} y(v_i)a_i\|^2.$$

which implies that $\mathrm{rk}(P_0) = \mathrm{rk}(V,d) = \frac{12\times 13}{2} - 1 = 77$.

It is useful to compare the above computation of $\mathrm{rk}(P)$ with the following computations using distances. Recall that $\mathrm{rk}(V(P_0),d)$ is equal to the dimension of the face of the hypermetric cone $HYP(V(P_0)) = HYP(V)$, where the distance d lies. The dimension of $HYP(V)$ is $N = \frac{|V|(|V|-1)}{2} = \frac{14\cdot 13}{2} = 91$.

As in Section 2, we obtain that, for every $w \in V = V(P_0)$, the equality (16) implies the following hypermetric equality

$$\sum_{v,v'\in V} y^w(v)y^w(v')d(v,v') = 0, \tag{17}$$

where $y^w(v) = y(v) + \delta_w$. It is easy to see that the 14 equalities (17) for 14 vertices $w \in V$ are mutually independent. In fact, these 14 equalities are equivalent to the 14 equalities (16) for the 14 vertices $u \in V$. The two equations (16) corresponding to two vertices $u, w \in V$ have only one common distance $d(u,w)$. The intersection of the corresponding 14 facets is a face of dimension $91 - 14 = 77$.

But, for every $u \in V$, the hypermetric space $(V - \{u\}, d)$ has rank $\frac{(|V-\{u\}|)(|V-\{u\})|-1)}{2} = 78$, which is greater than $\mathrm{rk}(V,d) = 77$.

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